In “How to Use Security Analysis to Improve Portfolio Selection”, The Journal of Business, Vol. 46, No. 1, p. 66-86, (1973) Jack L. Treynor and Fischer Black describe a model for active portfolio management. They show how to combine positions in individual securities on which an investor has active views with a passive (index) investment to build an expected-utility-maximizing portfolio of risky assets – which is, given the existence of a risk-free rate, the portfolio that maximizes the ratio of expected return to volatility – or as Treynor-Black call it: the Sharpe ratio.¹

The Froot-Stein² model discussed in the previous two posts is a framework to link risk management, capital budgeting and capital structure decisions in publicly traded institutions, and hence the model setting differs significantly from Treynor-Black. Nevertheless, as shown in this text, the Treynor-Black approach can be viewed as a special case of the capital budgeting decision as one element of the Froot-Stein framework – with the additional assumptions needed for this special case highlighting core differences between the models.

For the comparison with Treynor-Black the text describes the Froot-Stein approach with multiple investments, as well as the case with the bank having an active view on the market factor, both of which have not been discussed in this blog, yet.³

¹ See footnote 28 for more details. Note that this is the Sharpe ratio in “ex ante” terms, and used throughout by Treynor/Black and in this text in that sense – i.e. based on expected return and risk parameters, as opposed to the Sharpe ratio as performance measure based on realized return distributions. Treynor/Black further assess the impact of active views on portfolio diversification, describe how forecasts of active returns can be derived and develop further relationships between risk and return in efficient portfolios under their specific assumptions. Neither of which is discussed here.


See also the previous two posts in this blog. Symbols used in this post are the same as in the previous posts, unless otherwise stated.

Note that footnote 15 in the previous post stated that the appendix in the first post on the derivation of Froot-Stein proposition 1 and 2 contained redundancies. This is not correct, and therefore the appendix in the first post, will not be modified as stated in that footnote.

Further, while in the previous post the derivation of an optimality condition for the size of new exposures in the portfolio of the Froot-Stein bank (or other financial institution) was based on the Li/Ziemba approximation employing the Rubinstein measure of risk aversion, alternatively one could have applied results Rubinstein derived directly in a CAPM context – which are briefly described later in this post.

Also note that the first post on Froot-Stein in this blog and the derivation of their propositions 1 and 2 has been revised recently, as the description of the arbitrage trades if these propositions would not hold contained a mix-up of terms. For future versions of the previous two posts, the usage of the terms “CAPM” and “market portfolio” will be reviewed for future versions – for reasons given in this text – see footnote 6 and the concluding section. See also footnote 13, for a reference to motivation for using the term “market portfolio” by Treynor/Black – the motivation for the usage of CAPM and market portfolio in the previous posts was similar.

³ The case with an active view on the market is approached differently in Froot-Stein than briefly discussed here, this will be described in more detail in a later post.
The text concludes with insights by David Mayers and Ney O. Brito, to briefly describe the consequences of all investors in the market taking their non-tradable assets into account for their portfolio selection decision like the institution in the Froot-Stein model.

The security market models employed by the Treynor-Black investor and the Froot-Stein bank

Like Froot-Stein assume tradable risks are priced with a one factor model, Treynor-Black assume the investor employs a one-factor model (referred to as Sharpe’s Diagonal model) to form their expectations about expected security returns and risk. A difference between the two models is that the Diagonal Model explicitly assumes that the single factor is the only source of correlation between securities, while the pricing model for tradable securities in Froot-Stein does not need such an assumption (but it would still “work” with that assumption as well). In that sense, the model of the market of traded risks which the Treynor-Black investor applies is a special case of the model employed by the Froot-Stein bank.

Weights vs Exposure, Payoffs vs. Returns

A further, more technical difference is that in Froot-Stein the statistics considered refer to payoffs, while in Treynor-Black return numbers are analyzed. Also, correspondingly, in Treynor-Black the decision variables are the weights of the securities in the portfolio, while in Froot-Stein they are exposure sizes measured with in principle arbitrary units. However, the conventions can be easily reconciled. In Froot-Stein, there is another reason, which is that they explicitly consider non-tradable assets. But if an asset is strictly untradable, per definition it does not have a market value. Therefore there is no basis to relate future payoffs to when calculating a return number (however, if an asset is “untradable” in the sense that from the perspective of the bank that it is just not possible to trade it at a price the bank considers as fair, as it has different return expectations – a case discussed by Froot-Stein and described later in this text – this constraint does not apply that strictly, as in the case with interpreting the untradable asset as a forward on an asset which has a market value). In general, for highly levered portfolios, weights of individual legs of financed positions, which might become very large, might for practical reasons not be the most convenient choice – this will be reviewed in a later post.

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4 See footnote 1 in the previous post and the references given there for more details on the pricing equation, where it was stated that it can be derived from the CAPM SML. An additional reason Froot-Stein don’t refer to this model explicitly as “CAPM” can probably be found in Mossin 1966, Equilibrium in a Capital Asset Market, Econometrica, Vol. 34, No. 4, Oct. 1966, p.779: “If one or more individuals do not behave rationally, the whole foundation of the analysis is destroyed, and the concept of equilibrium and hence also of the market line, becomes meaningless”. As the CAPM assumes investors with homogeneous expectations regarding return distributions, deviating from this homogenous expectations – which at least the trading desk of the Froot-Stein bank does, could hence be viewed as deviating from rational behaviour under the CAPM assumptions. Therefore, as the quote by Mossin emphasizes, strictly speaking a crucial assumption of the basic CAPM is violated. The resulting consequence for the pricing equation is discussed in the concluding section of this text.

5 The term diagonal model for the “single index” model or “market model” was introduced by William F. Sharpe, A Simplified Model for Portfolio Analysis, Management Science, Vol. 9, No. 2 Jan. 1963, p.277-293. The model was introduced more briefly by Harry Markowitz – see the specific reference and additional details in the post on CAPM and the return generating process in this blog. Treynor/Black explicitly use the market portfolio as the single index (while as Sharpe noted it could be any other factor deemed as single most important influence – e.g. GNP or a price index) which implies an approximation - see the next footnote.

6 That the market portfolio is the only source of correlation between traded securities, is not a necessary assumption of the CAPM nor the Froot-Stein pricing equation, actually this assumption contradicts the assumption that the “independent returns” of all securities are absolutely uncorrelated with each other, hence Treynor/Black highlight that assumption can only hold approximately, given that all securities are a component of the market portfolio. See Treynor/Black, page 69-69, and (for example) also Sharpe’s Nobel lecture, page 320, footnote 16.

7 Besides the reasons given for using payoffs in Froot-Stein, there is another reason, which is that they explicitly consider non-tradable assets. But if an asset is strictly untradable, per definition it does not have a market value. Therefore there is no basis to relate future payoffs to when calculating a return number (however, if an asset is “untradable” in the sense that from the perspective of the bank that it is just not possible to trade it at a price the bank considers as fair, as it has different return expectations – a case discussed by Froot-Stein and described later in this text – this constraint does not apply that strictly, as in the case with interpreting the untradable asset as a forward on an asset which has a market value). In general, for highly levered portfolios, weights of individual legs of financed positions, which might become very large, might for practical reasons not be the most convenient choice – this will be reviewed in a later post.
Stein, positions explicitly do not require any cash outlay, i.e. they are financed or forward positions. Hence, the previous posts used the terms “exposure” or “size” instead of weight – as the latter is zero for no-cash outlay positions. Alternatively, one could split the zero-cash-outlay position into two legs, an investment into the underlying asset, and a loan to finance that investment – with the combination of both providing a payoff equal to a forward contract. Further, as Froot-Stein point out, one could interpret the payoffs in the Froot-Stein model as payoffs per dollar of the value of the underlying asset (more precisely: payoffs of an underlying asset with a price of 1), and the payoffs as displayed in Froot-Stein would then be returns of the underlying assets after subtracting the financing costs (assuming financing with the risk-free rate)\(^8\), from which the hedging costs for systematic risk are subtracted where the hedge is performed (or simulated). The return after financing and hedging costs is then exactly the same as the “independent return” in Treynor-Black, i.e. the return part that is not correlated with the factor (e.g. a stock market index) of the Diagonal Model.\(^9\)

In the previous posts, and in the original Froot-Stein notation, the payoff of an exposure to a new product was given as the product of the size of the exposure and the payoff associated with the new product. Hence it was not necessary to give the units of the size measure, nor the unit this payoff was referring to. One could essentially develop an arbitrary definition of the units of either, as long as the product of both gives the payoff resulting from the new exposure. For example, if \(N_z\) is interpreted as payoff per dollar notional exposure of the forward (i.e. the return of the forward’s underlying asset), \(\alpha\) is the number of dollars notional exposure. To establish full compatibility with Treynor-Black notation, one could then divide \(\alpha\) by the total bank capital \(K\), to get the weight this forward’s underlying asset has essentially in the portfolio. To assess the impact of this change on the previous results, recall from the previous post, that the Froot-Stein optimality condition is:

\[
\mathbf{I.} \quad \text{cov}(P_w, w_\alpha) + E P_w E w_\alpha = 0
\]

If instead of payoffs, returns are used, and the payoffs are “recovered” by multiplying the returns with \(K\), then instead of

\[
E w_\alpha = \mu_{NH}
\]

as in the previous post, now the following holds:

\[
E w_\alpha = K\mu_{NH}
\]

With \(\mu_{NH}\) now being the expected return of a new product’s underlying asset (after subtracting the risk free rate and the premium for the product’s “amount” of systematic risk).

Also, instead of

\(^8\) This implies, that at least in the period before investing in the asset with concave payoff (see the first post in this blog with details on the Froot-Stein timeline), the bank has access to financing at the risk-free rate – more details on this point will be provided in a later post.

\(^9\) Treynor/Black, page 68, 69, equation 1. See the previous post for more details on simulation of hedges and the subtraction of hedging costs.
\( \text{cov}(P_w, w_{\alpha}) = \text{cov}(P_w, e_{NH}) = EP_{w \alpha} \text{cov}(w, e_{NH}) = EP_{w \alpha} \sigma_{w NH} \)

now the expression for the covariance, after applying the Rubinstein-Stein Lemma becomes:\(^1\)

\( \text{cov}(P_w, w_{\alpha}) = K \text{cov}(P_w, e_{NH}) = KEP_{w \alpha} \text{cov}(w, e_{NH}) = K^2 EP_{w \alpha} \sigma_{x NH} \)

Where \( x \) indicates the total portfolio return \( w/K \)-1.

The optimality condition changes then to:

\( K^2 EP_{w \alpha} \sigma_{x NH} + EP_{w} K \mu_{NH} = 0 \)

And can be simplified to:

\[ \mu_{NH} = -K \frac{EP_{w \alpha}}{EP_{w}} \sigma_{x NH} = KG \sigma_{x NH} \]

II.

\( KG \), in the following expressed as \( G_K \), is in the literature referred to as Rubinstein’s measure of relative risk aversion.\(^2\)

**Active views on tradable assets’ idiosyncratic return components**

In Treynor-Black ’73, the perspective is that of an investor who engages in security analysis to generate “special insights”, so that they might consider certain securities as “overpriced” or “underpriced”\(^3\). In the terms of the Diagonal Model employed by the individual investor, this means that for such securities, the investor expects a value of the residual (independent) return different to zero.\(^4\) Treynor-Black refer to the value of the independent return that is expected by the investor as “appraisal premium”.

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\(^1\) Note that in this post covariances are labelled for example as \( \text{cov}(w, e_{NH}) \) or \( \sigma_{w NH} \) interchangeably. While all symbols are explained in the text, in a revised version a list of symbols used will be added to the text (and a list of references – currently all references are given in footnotes to this text – in some cases indirectly when the reference is to a previous post where the citation of the original reference can be found).

\(^2\) Note that in previous posts various sources were given for “Stein’s Lemma” – however written in the manner closest to used here, it can be found in Rubinstein, The Valuation of Uncertain Income Streams and the Pricing of Options, The Bell Journal of Economics, Vol. 7, No. 2 (Autumn, 1976), pp. 407-425, p. 420 – see also the references given there in footnote 20. There Rubinstein also notes that the first publication that contained this lemma was by himself in 1971 but that it was independently discovered by Stein in 1973 – hence in this text it is labelled “Rubinstein-Stein Lemma.”

\(^3\) See page 275 of the paper by Li and Ziemba mentioned in the previous post. Alternatively, one could change the utility/final payoff function with the portfolio returns being the independent variable and use the Rubinstein measure of absolute risk aversion \( G \) instead of \( G_K \).

\(^4\) One could also say, the investor “translates” the expectations of an otherwise homogenous market in a state of CAPM equilibrium into the Diagonal Model to express relationships between expected security returns and risk parameters, and “attempts to trade profitably on the difference between his expectations and those of a monolithic market” (Treynor/Black page 68).
This perspective is similar to that of a trading desk as described by Froot-Stein.\textsuperscript{15} Recall that under Froot-Stein proposition 2, the bank would hedge all tradable risks completely. However, one assumption to derive this proposition was that the bank considers all tradable risks as “fairly priced”. And as Froot-Stein note when discussing the application of their model for a trading desk: “...being a trading desk by definition requires intentionally assuming certain exposures. Ostensibly, these exposures are justified by the desk’s ability to earn a positive return on average, even after adjusting for market-wide risk factors. The presence of such positive, albeit subjective, risk-adjusted returns, makes such exposures non-tradable in our sense.”\textsuperscript{16}

As mentioned earlier, the pricing model for tradable securities employed in the Froot-Stein framework can be viewed as a generalization of the Diagonal model, and the Treynor-Black “independent return” corresponds to the return after subtracting the cost of hedging systematic risk in Froot-Stein, in that sense what they refer to in this quote as “risk-adjusted return” corresponds to the Treynor-Black appraisal premium.\textsuperscript{17}

And like a new investment in the Froot-Stein model under the “perfect hedging proposition” is assessed simulating a complete hedge of systematic risk\textsuperscript{18}, in the Treynor-Black approach the investor follows a three stage process, and in the first stage takes “positions in securities 1, ..., n” purely on the basis of expected independent return [appraisal premium] and [independent] variance [and] The resulting exposure to [systematic] market risk is disregarded [at this stage].\textsuperscript{19}

**Multiple simultaneous decisions**

Treynor-Black model an investor constructing a portfolio out of various securities. For a comparison with Froot-Stein, one must therefore review their analysis of multiple investments at the same time, which has not yet been discussed in this blog (in the previous post, (only) the Froot-Stein “base case” was discussed, with the bank deciding about only one new exposure).

The core difference to the case with only one new exposure is, that in the case with several simultaneous decisions about new positions, not only the covariance of a new position with the existing portfolio, but also its covariance with all other new positions has to be taken into account.\textsuperscript{20}

Recall the general optimality condition which via the covariance of the new product with the existing portfolio determines the optimal weight (equation II above).\textsuperscript{21}

\textsuperscript{15} Froot-Stein page 72.
\textsuperscript{16} Ibid.
\textsuperscript{17} Note that Froot-Stein describe these returns as earned “on average”. In the context here, this is understood as an “ex ante” average, i.e. a cross sectional average across possible future states of the world (in an Arrow/Debreu sense), not as an average of realized historical returns.
\textsuperscript{18} See the previous two posts.
\textsuperscript{19} Treynor/Black page 72. Amendments in square brackets by the author of this text
\textsuperscript{20} Note that in the Diagonal Model employed by the Treynor/Black investor the covariances are zero - this special case will be discussed later in the text.
\[ \mu_{NH} = -K \frac{E_{PW}}{E_{P}} \sigma_{sNH} = G_K \sigma_{sNH} \]

In the case of a single investment, the covariance of the new product’s underlying asset’s return (after hedging systematic risk) with the total portfolio return (after hedging systematic risk) is:

\[ \sigma_{x, NH} = \sigma_{pNH} + \alpha \sigma_{NH}^2, \]

where \( p \) now represents the existing portfolio return after hedging systematic risk (as \( x=p+NH \)). With multiple new positions entered at the same time, and hence \( x = p + \sum NH_i \), the covariance of any new product with the total portfolio becomes:

\[ \sigma_{x, NH} = \sigma_{pNH} + \sum_{j} \alpha_j \sigma_{NH,NH_j}. \]

Following now the original Froot-Stein notation more closely and writing \( \pi_i \) for \( \mu_{NH, i} \), an equation for the optimal weight can be built by plugging this expression into the optimality condition:

III. \[ \pi_i = G_K \left[ \sigma_{pNH} + \sum_{j} \alpha_j \sigma_{NH,NH_j} \right] \]

Slightly rearranging gives:

IV. \[ \frac{\pi_i - G_K \sigma_{pNH}}{G_K} = \sum_{j} \alpha_j \sigma_{NH,NH_j} \]

The right part of this set of \( n \) equations (one for each new product \( i \)) is the covariance matrix of any new product with the total portfolio of new exposures multiplied with the vector of weights of the new products in the total portfolio.\(^{22}\) In vector/matrix notation, it is the product \( (\alpha^* \Omega) \) of the vector of new (optimal) exposure sizes \( \alpha^* \) with the covariance matrix \( \Omega \) of the payoffs of the new products the bank enters positions in.

To follow the Froot-Stein notation, write for the numerator of the left side of the equation in matrix notation:

\[ \pi - G_K C_{NP} \]

With \( \pi \) and \( C_{NP} \) each being a (column) vector containing \( \pi_i \) and \( \sigma_{pNH} \) respectively for each new product \( i \).

Multiplying both sides of the equation with (the scalar) \( G_K \) results in following expression for the equation set:

\[ \pi - G_K C_{NP} = G_K \alpha^* \Omega \]

\(^{21}\) See previous post, page 7. Unfortunately, instead of \( \mu_{NH} \) in the “restated optimality condition” on page 7 \( \mu_H \) was written – this will be corrected in a revised version of that post.

\(^{22}\) I.e. this is the covariance of the return of new product \( i \) with the return of total portfolio of new products.
Multiplying both sides with the inverse $\Omega^{-1}$ of the covariance matrix of the new products’ payoffs and dividing by $G_K$ results in:

$$\alpha^* = \Omega^{-1}(\pi - G_K C_{NP})/G_K$$

This is almost identical to Froot-Stein equation 12, with the exception that $G_K$ is used here instead of $G$, and returns instead of payoffs and weights instead of absolute exposure sizes. Note that as Froot-Stein point out, the elements in their vector $\pi - GC_{NP}$ (with payoff variables) can be interpreted as the “net payoffs” of the new products, where “net payoff” is a product’s payoff after subtracting a premium to adjust for the product’s exposure to systematic risk, as well as it’s covariance with the existing portfolio (and after subtracting financing costs at the risk-free rate) - analogously $\pi - G_K C_{NP}$ (as above with return notation) can be interpreted as net returns after deducting the risk free rate and these two risk premia expressed as returns. Note that both risk premia for each new position are independent of other new positions to be taken on at the same time.

No existing portfolio of untradeable exposures at the time of the decision about new exposures

Treynor-Black do not explicitly mention the existing investor portfolio at the point in time when the investor performs their portfolio selection. As in the Treynor-Black model all assets considered are assumed to be perfectly tradable, there is no need (nor use) for such a distinction between the existing portfolio and new products, the investor can select the optimal weights for every asset simultaneously, treating any positions currently held in the portfolio equally to any new positions. I.e. it is not necessary to consider the covariance with any possibly already existing portfolio separately. In other words, $\sigma_{pNH}$ can be set to zero for every product $i$ and any existing position can be included in the covariance matrix like all new positions.

Equation III hence simplifies to:

$$\pi_i = G_K \sum_j \alpha_j \sigma_{NH,NH_i} = G_K \sigma_{ix}$$

This is essentially an equation derived by Rubinstein for an investor maximizing their expected utility, only that Rubinstein’s equation was based on return distributions before hedging systematic risk.

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23 See page 614 in: A Comparative Statics Analysis of Risk Premiums, Mark E. Rubinstein, The Journal of Business, Vol. 46, No. 4 (Oct., 1973), pp. 605-615. In original notation equation 6 on that page is: $$\left[ E(1+n_i) - (1+r) \right] \frac{1}{\theta} = \text{cov}\left(1+n_i, W_0\left(1+r_p\right)\right)$$ for all $i$, with $\theta = -\frac{EU}{EU}$ as risk aversion parameter introduced by Rubinstein on page 606 and 614, and $p$ indicating the total investor portfolio. This can of course easily be rearranged to express the relationship in return terms: $$\left[1 + \mu_i - 1\right] \frac{1}{\theta} = \text{cov}\left(1+n_i, W_0 + W_0\varphi_p\right) = W_0\sigma_{ip}$$ or in the notation used here (replacing $W$ with $K$ and $\theta$ with $G$, and writing $\pi_i$ for $\pi_i - r$ , and $x$ instead of $p$). That returns after hedging systematic risk are used here does not change the general relationships
Active view on systematic risk and one core difference between a private investor and a publicly traded bank

In Treynor-Black, an optimal investment into the market portfolio\(^{24}\) (which is to be added “in a second stage”) is derived as well. As with other traded risks, under Froot-Stein proposition two, if the bank considers the market portfolio as fairly priced its exposure to the latter would be zero. Recall however, that in the case that the bank does not consider the market portfolio as fairly priced, i.e. the bank has an active view on the market, Froot-Stein explicitly also allow for such an exposure.

How to deal with this case is discussed more comprehensively in a later post. However, in the following the assumptions needed to derive Treynor-Black as a special case of the Froot-Stein capital budgeting stage are explained briefly – and why these assumptions highlight core differences between the models.

In Froot-Stein the market portfolio is like in Treynor-Black assumed to be perfectly tradable. Hence, the aggregate exposure to the market portfolio can be viewed as one more of the “new” products, with all the other new products being assessed after simulating a perfect hedge of their systematic risk, and the market portfolio as carrying only systematic risk being uncorrelated with each of the other new products, as well as with the existing portfolio of untraded assets.\(^{25}\)

However, as will be discussed in more detail in a later post, if a bank has an active view on the overall market, it will apply a different capital budgeting approach than the one used for untradeable assets and tradable but in the view of the bank unfairly priced assets (while in the absence of such an active view it will take no exposure to the market – see the earlier post on Froot-Stein propositions 1 and 2). This is a core difference to the Treynor-Black model. To see the Treynor-Black approach as a special case of the capital budgeting part of the Froot-Stein model, requires hence either that the market risk premium is assumed to be zero, so that the investor in Treynor-Black would not want to hold exposure to the market portfolio, or that the bank is held by a private investor and constitutes the total wealth of that investor (at least until \(P(w)\) is realized). Both assumptions would be inappropriate. The first one would be a strong additional restriction of the Treynor-Black model, while the second assumption removes main elements from the Froot-Stein model. Nevertheless, the additional generalizations provided by Froot-Stein, i.e. allowing for assets that are untradeable at prices subjectively viewed as fair and allowing for non-zero correlations between assets after stripping off systematic risk, make the capital budgeting part of the model attractive as a general framework that allows to “switch” between a private investor and a traded bank.

\(^{24}\) In the following, up to the concluding section, the term “market portfolio” will be used as in Treynor/Black and the previous posts. Note that Froot-Stein use the term market factor instead. This will be further commented in the concluding section. See also footnote 2.

\(^{25}\) In Treynor/Black this is expressed in equations 7-13 on page 71, where the market portfolio is included in the portfolio optimization like any other asset.
Uncorrelated specific risks of new products

Treynor-Black assume that the return part of every security that is not explained by the security’s covariance with the market portfolio (the “independent” part of its return), is uncorrelated with every other security’s independent return part. If one analogously assumes correlations of zero between the positions’ payoffs (after hedging) in Froot-Stein, the covariance of a position with all other positions in the portfolio become zero, and its covariance with the total portfolio is just the position’s weight times the underlying asset’s variance. Hence, equation V. simplifies to:

\[ \pi_i = G_K \alpha_i^* \sigma_{NH}^2 \]

And the optimal weight of the new position is:

\[ \alpha_i^* = \frac{1}{G_K} \frac{\pi_i}{\sigma_{NH}^2} \]

This set of equations is similar to equation set 10 in Treynor-Black, with the only difference being that instead of \( \frac{1}{G} \), the Lagrange multiplier (“undetermined multiplier”) \( \lambda \) is used in Treynor-Black (Treynor-Black refer to this \( \lambda \) later as the investor’s “attitude towards risk bearing”, without elaborating further). Treynor-Black derive the following expression for the Lagrange constant:

\[ \frac{\sigma_x^2}{\pi_x} = \lambda \]

This can be derived from the more general relationship in efficient portfolios.
VI. \( \pi_i = \frac{\pi_x}{\sigma^2_x} \sigma_{ix} \)

As mentioned above, Rubinstein has shown that in efficient portfolios also the following holds (see above equation V):

\[ \pi_i = G_k \sigma_{ix} \]

Hence it must be that:

\[ G_k = \frac{\pi_x}{\sigma_x^2} \]

To derive the Treynor-Black equation, note that in the special case of uncorrelated assets, \( \sigma_{ix} \) simplifies to \( \alpha_i \sigma_i^2 \) – hence, substituting in equation VI and rearranging gives:

\[ \frac{\pi_i \sigma_x^2}{\sigma_{ix}^2} = \alpha_i \]

Which is Treynor-Black equation 13 for the optimum position in asset \( i \), for \( i = 1, ..., n \), where as explained in the previous section, the aggregate exposure to the market portfolio is included as one of these assets, and all other assets’ parameters are after hedging systematic risk.\(^{29}\)

\(^{29}\)Treynor/Black derive this expression by minimizing total portfolio variance for a given total portfolio excess return (return minus risk-free rate, but of course before hedging systematic risk...) to return variance is sometimes referred to as “market price of risk” (see Rubinstein ’73, (complete reference given in footnote 23) page 611 for an overview of various definitions of the “market price of risk”). Mossin was critical of that term, suggesting instead “price of risk reduction”, arguing that one wouldn’t use the term “price of garbage” for a sanitation fee, either (Mossin, 1966, p. 781 – see footnote 4 for complete citation) – it would be interesting to know what Mossin would think about the term “risk appetite”, which gained popularity amongst practitioners recently as a synonym for the degree of risk aversion or risk tolerance.

Finance, March, 1968, p. 29-40 (see in particular p. 35 and 36). In a CAPM context, for the market portfolio, the ratio of excess return (return minus risk-free rate, but of course before hedging systematic risk...) to return variance is sometimes referred to as “market price of risk” (see Rubinstein ’73, (complete reference given in footnote 23) page 611 for an overview of various definitions of the “market price of risk”). Mossin was critical of that term, suggesting instead “price of risk reduction”, arguing that one wouldn’t use the term “price of garbage” for a sanitation fee, either (Mossin, 1966, p. 781 – see footnote 4 for complete citation) – it would be interesting to know what Mossin would think about the term “risk appetite”, which gained popularity amongst practitioners recently as a synonym for the degree of risk aversion or risk tolerance.
Market portfolio vs. market factor

A similar relationship like equation VI can be developed with the existence of non-traded assets in the current portfolio. David Mayers developed the following equation for a tradable asset in a market with homogenous expectations and the existence of nonmarketable assets in investor portfolios:

$$
\mu_i = r + \frac{\mu_M - r}{M\sigma_M^2 + \sigma_{MH}} [M\sigma_{IM} + \sigma_{iH}]
$$

Where the index $H$ indicates the total future dollar amount to be received by all investors from nonmarketable assets, $M$ is the total (current) value of the market portfolio, or, where $M$ is used as index (subscript) indicates the return of the market portfolio, and $i$ indicates a single asset’s return (before subtracting the risk free rate or simulating any hedge of tradable risk), and $r$ is the risk-free rate as before.

Similar to the Froot-Stein and the Treynor-Black approach, one could first simulate removing any correlation with the market portfolio from $H$ by combining $H$ with a position in $M$ that offsets any exposure to $M$ which $H$ might carry. The optimal exposure to $M$ is then as above derived completely separately. In the following, $H$ hence stands for non-marketable assets that have been hedged in that sense and don’t contain any exposure to $M$ anymore – i.e. the correlation of $H$ and $M$ is 0. This results in a modified version of Mayers’ equation:

$$
\mu_i = r + \frac{\mu_M - r}{K\sigma_M^2} [K\sigma_{IM} + \sigma_{iH}]
$$

VII.

The equation $\pi - GC_{NP} = G\alpha^* \Omega$ is divided by $G$ and rearranged to $0 = \alpha^* \Omega - \frac{1}{G}(\pi - GC_{NP})$, one gets an equation very similar to Merton’s equation 34a for the case with a risk-free asset (which is consistent with the Froot-Stein assumption for this period, see footnote 8) with the only differences being that in Froot-Stein instead of the Lagrange constant, $1/G$ is used, and the two risk premia are subtracted from the excess return. Equation V on the other hand, was derived by Rubinstein under the same assumptions directly by maximizing utility, using a Taylor series expansion (while the Froot-Stein derivation of equation III – of which equation V is a simplification for the special case without existing untradable assets – is established via the definition of covariance and the Rubinstein-Stein Lemma, which is based on a normal distribution assumption), under the assumption of a concave “utility” function. As Merton’s, Rubinstein’s and Lintner’s assumptions are equivalent, therefore for the Lagrange constant the following holds:

$$
\frac{1}{\lambda} = G_K = \frac{\pi_p}{\sigma_p^2}
$$

which explains why Treynor/Black as mentioned before refer to the Lagrange constant as “the investor’s attitude towards risk bearing”, page 73.

David Mayers, “Non-marketable Assets and Capital Market Equilibrium under Uncertainty” in Studies in the Theory of Capital Markets, ed. M.C. Jensen (New York, Praeger Publishers, 1972). Note that Mayers, like Brito, had in particular “human capital” or occupational assets in mind, which are objectively untradable. In Froot-Stein, as described earlier in this post, untradable assets are also those which are only tradable at terms the bank does not consider as fair.

This equation is a slightly modified version of equation 13, p. 262 in: Nonmarketable Assets and the Determination of Capital Asset Prices in the Absence of a Riskless Asset, David Mayers, The Journal of Business, Vol. 46, No. 2 (Apr., 1973), pp. 258-267. Mayers states in this source, that the equation given in the source mentioned in footnote 30 is the same, except that the risk-free rate replaces the consumer’s marginal rate of substitution of future expected returns for current consumption, hence this replacement has been performed here to get to the equation given above.
Now one can further simplify the analysis, by assuming that not only exposure to \( M \) but also to other sources of correlation with tradable assets has been removed from \( H \), by taking offsetting positions in these assets – with determining the optimal aggregate exposure to such assets in a separate step. In that case, equation VI simplifies further, to:

\[
\mu_i = r + \frac{\mu_M - r}{\sigma_M^2} \sigma_{iM}
\]

VIII.

This would be the basic CAPM pricing equation, i.e. without the existence of untradeable assets.\(^{32}\) However, as pointed out by Ney O. Brito, investors behaving in this manner results in the market portfolio not necessarily being an efficient portfolio anymore.\(^{33}\) Brito points out, that the minimum variance portfolio in the presence of non-marketable (untraded) assets is a combination of the investors’ nonmarketable portfolio and “corrective” positions to strip off any covariance with any marketable asset or portfolio.\(^{34}\)\(^{35}\) As Brito describes, if investors apply the mean-variance approach (e.g. if they are expected utility maximizers and the Tobin separation applies), and want to increase the expected return of their portfolio with portfolio variance increasing as little as possible, they will add units of the Sharpe Ratio-maximizing portfolio of marketable assets to their specific minimum variance portfolio including their nonmarketable assets.\(^{36}\) Brito argues further, that under homogeneous

\(^{32}\) See the sources given in footnote 28.


\(^{34}\) The variance of a portfolio containing an untradeable asset \( u \) with a purely untradeable return component \( h \) but nonzero covariances with returns of tradable assets indexed \( i \) and \( j \), and corrective positions in tradable assets can be calculated as:

\[
\sigma_p^2 = \sigma_u^2 + \sum_i \sum_j w_i w_j \sigma_{ij} \text{ with aggregate exposures to tradable assets labelled } w_i. \text{ It is clear that in the minimum variance portfolio the double sum on the right is zero, which is the case if all sensitivities of the aggregate portfolio to tradable assets, } w_i, \text{ are zero as well. And as the covariance of the aggregate of the untradeable asset } u \text{ and corrective positions with any tradable asset or portfolio of tradable assets (only) equals: } \sigma_{u+c,T} = \sum_i \sum_j w_i w_j \sigma_{ij}, \text{ with } w_j \text{ now being the exposure to the tradable asset } j \text{ in the portfolio of tradable assets only, and as } w_j \text{ are all zero, the minimum variance portfolio is uncorrelated with any tradable asset or portfolio of tradable assets (instead of assets, one could also define risk factors, the weights would then be replaced with factor sensitivities). In other words, the covariance of the portfolio of corrective positions with any marketable asset or portfolio equals the covariance of the investor’s nonmarketable wealth with that portfolio times minus one (see Brito, p. 1111).}

\(^{35}\) Depending on the covariance structure of securities, there might be other portfolios that have the same minimum variance, more generally: any combination of expected return and variance on the efficient frontier could theoretically represent more than one portfolio, Sharpe, 1964, (see footnote 28 for full citation) page 435 and John Lintner, Security Prices, Risk and Maximal Gains from Diversification, The Journal of Finance, Vol. 20, No. 4, Dec. 1965, p. 597, footnote 15. This does however not change Brito’s general reasoning or core results.

\(^{36}\) See Brito p.1110 for the general reasoning and p.1118 for the case with a riskless asset which is relevant here. Neither Brito nor Mayers discuss the case of a investors having to determine the exposure to a new untradeable asset – as they focus on human capital (occupational assets). However, the approach described by Brito implies a method for dealing with a new untradeable asset: similar to Froot-Stein and Treynor Black, the new asset is assessed after simulating a hedge of all tradable risks. Then the optimal allocation to this new asset and a new total allocation to the Sharpe-ratio-maximizing portfolio of tradable assets (which is then again uncorrelated with the aggregate of untradeable assets after hedging systematic risk) are determined – as in the Treynor-Black approach (see the section on multiple simultaneous decisions and footnote 25 above). For the untradeable component of the new asset, the investor would require a risk premium – as in Froot-Stein – however this premium would not have any direct impact on the market risk premium, nevertheless, the investor might change the allocation.
expectations, the basis for the pricing equation for marketable assets will hence be this Sharpe-Ratio maximizing portfolio of marketable assets. The total market portfolio on the other hand, now consists of the Sharpe-ratio maximizing portfolio of traded assets, plus the corrective positions investors have taken to create their specific minimum-variance portfolios. Different to the CAPM without considering marketable assets, the market portfolio therefore is not (necessarily) an efficient portfolio anymore, and hence not the basis for the pricing equation. Both equation VI. and equation VII. are hence valid if the market portfolio $M$ is replaced with the efficient portfolio of marketable assets.

As mentioned repeatedly in this blog, Froot-Stein don’t refer to the pricing equation for the bank shares as “CAPM” (they use the expression “one factor model”). Further, neither do they refer to $M$ as the market portfolio (they use the term “market factor” instead). This last section of this post might explain this choice of terminology. Recall footnote 4 with the reference to Mossin, and that as the bank takes its untradeable exposures into account it constitutes one market participant deviating from the CAPM assumption set. However, if the market factor $M$ in the Froot-Stein pricing equation for tradable assets is not the market portfolio, but the Sharpe-Ratio-maximizing portfolio of marketable assets, then, as Brito has shown, the pricing equation VII is valid even, or in particular if, every market participant behaves like the bank in the sense that they take their untradeable assets into account for their portfolio selection. Hence, the Froot-Stein pricing equation, which is based on equation VII (see the first post on Froot-Stein ’98 in this blog) is consistent with the bank’s capital budgeting approach.

Note however, that in the case of active views on tradable assets, the assumption of homogeneous expectations regarding risk and return of tradable assets is violated – so that a further extension of the framework is needed, if one wants to allow that the overall market behaves similar to the Froot-Stein bank. This is discussed in a later post.

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37 For the derivation of a pricing equation from the return form shown above see the first post on Froot-Stein in this blog.
38 This insight highlighted by Brito can be seen as a basis for a critique of the usage of market-capitalization weighted indices being popular as benchmarks in institutional asset management.