Consider a mean-variance-optimizing investor and assume there are \( n \) risky assets (indexed with \( i \)).

**Naïve diversification**

**Zero Correlations between non-identical assets**

Assume the only thing the investor knows is that correlations between any two different assets are zero. They have no information at all about assets’ expected returns or volatilities. Hence, the investor treats every asset equally, as if all \( n \) assets (indexed \( i \) or \( j \)) had the same expected return and volatility \( \sigma \):\[ \sigma_j = \sigma_j = \sigma \quad \forall \ i, j \]

If all assets have the same expected return, and with full allocation of wealth, the expected portfolio return will be the same for any portfolio formed of these assets. Therefore, the best the investor can do is to minimize the portfolio variance.

As the investor (due to lack of better knowledge) assumes all assets have the same volatility, and non-identical assets are uncorrelated, the optimal portfolio is the naively diversified portfolio, i.e. a portfolio that contains all available assets and in which each asset has the same weight. This can be shown in various ways, below is a “calculus-free” proof:

With naïve diversification, all weights \( (w_i) \) equal \( 1/n \) and with zero correlations between non-identical assets the portfolio variance is:

\[
\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 = n - \frac{1}{n^2} \sigma_i^2 = \frac{1}{n} \sigma_i^2
\]

Increasing the weight of any asset slightly and reducing the weight of any other asset \( (i=2) \) by the same amount leads to a new portfolio with a variance as follows:

\[1^1\text{ The following attempts to provide an intuitive systematic of these concepts, thereby expanding on thoughts given on page 5 of: Risk Parity versus Mean-Variance: It’s all in the Views, Daniel Haesen, Winfried G. Hallerbach, Thijs Markwat, Roderick Molenaar, December 2014.}
\[ \sigma^2_{\text{new}} = \sum_{i=3}^{N} w_i^2 \sigma_i^2 + (w_1 + x)^2 \sigma_1^2 + (w_2 - x)^2 \sigma_2^2 = \]

\[ \sigma^2_{\text{p}} - w_1^2 \sigma_1^2 - w_2^2 \sigma_2^2 + (w_1 + x)^2 \sigma_1^2 + (w_2 - x)^2 \sigma_2^2 = \]

The difference between the variance of the naïvely diversified portfolio and the new portfolio is:

\[ \Delta \sigma^2 = \sigma^2_{\text{new}} - \sigma^2_{\text{p}} = \]

\[ -w_1^2 \sigma_1^2 - w_2^2 \sigma_2^2 + (w_1 + x)^2 \sigma_1^2 + (w_2 - x)^2 \sigma_2^2 = \]

\[ \left( w_1^2 + 2w_1x + x^2 \right) \sigma_1^2 + \left( w_2^2 - 2w_2x + x^2 \right) \sigma_2^2 - \left( w_1^2 + w_2^2 \right) \sigma_1^2 = \]

\[ \left( 2w_1x + x^2 - 2w_2x + x^2 \right) \sigma_1^2 = \]

\[ 2(w_1 + x - w_2)x \sigma_1^2 = 2 \left( \frac{1}{n} + x - \frac{1}{n} \right) x \sigma_1^2 = \]

\[ 2x^2 \sigma_1^2 \]

which is always positive. So any deviation from the naïvely diversified portfolio increases the portfolio variance, i.e. if all correlations between non-identical assets are zero, and all assets’ volatilities are equal, the naïvely diversified portfolio is the minimum variance portfolio.²

If the uniform correlation between non-identical assets is not zero:

The general formula for the portfolio variance can be written as:

\[ \sigma^2_{\text{p}} = \sum_{i} w_i^2 \sigma_i^2 + \sum_{i \neq j} w_i w_j \sigma_i \sigma_j \rho_{ij} \]

With the same value for the correlation between all non-identical assets, and with identical volatilities for all assets, this simplifies to:³

\[ \sigma^2_{\text{p}} = \sum_{i} w_i^2 \sigma_i^2 + \rho \sigma^2 \sum_{i \neq j} w_i w_j = \sigma^2_{\text{p}} \sum_{i} w_i^2 + \rho \sigma^2 \sum_{i \neq j} w_i w_j \]

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² As the last equation shows, the increase in variance would grow with \( x \). Additional reallocations could be modelled subsequently (changing the starting value of the sum from 3 to 5 for the 2nd reallocation, then to 7 and so forth), to show that they would lead to further increases in variance.

³ There are restrictions on the values of a uniform correlation coefficient; simply put it cannot be too negative, for the covariance matrix to be consistent.
Because

\[ \sum_{j \neq i} w_j = (1 - w_i) \]

The equation for portfolio variance simplifies further to:

\[
\sigma_p^2 = \sigma^2 \sum_i w_i^2 + \rho \sigma^2 \sum_i w_i (1 - w_i) = \sigma^2 \sum_i w_i^2 + \rho \sigma^2 \sum_i (w_i - w_i^2) = \]

\[
\sigma^2 \sum_i w_i^2 + \rho \sigma^2 \left( \sum_i w_i - \sum_i w_i^2 \right) = \]

\[
\sigma^2 \sum_i w_i^2 (1 - \rho) + \rho \sigma^2 \]

As correlation and volatility are constant and the term in brackets is always at least zero, minimizing this expression corresponds to minimizing:

\[
\sigma^2 \sum_i w_i^2 \]

which is the portfolio variance in the case with zero correlations between all non-identical assets discussed above. So also with a non-zero, but uniform correlation coefficient for all pairs of assets, and equal asset volatilities, the naively diversified portfolio is the minimum-variance portfolio.

**Risk parity type 1 – creating positions with equal volatilities**

Imagine now, that the investor still believes that the correlation coefficient is the same for all pairs of assets, and they still have no precise information about expected returns. But now they know the different, individual volatilities of all assets. Further, lacking more precise information, the investor believes that higher asset risk would come with a higher risk premium; more precisely they assume the ratio of risk premium to volatility is the same for each asset.

Combining each of the \( n \) assets in the market with a loan, the investor could create \( n \) positions, such that these leveraged positions’ volatilities were all equal. The correlations will not be affected by this exercise, i.e. the result is a set of investments with indistinguishable volatilities, uniform correlation between different investments, and due to lack of better information assumed-to-be-equal expected returns. Recall that this was the starting point for the naïve diversification case.

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5 Risk premium = expected excess returns, i.e. expected returns minus risk-free rate.

6 The term asset is used here in a broad sense. An asset could for example be a highly diversified index fund representing an asset class.
Risk parity type I as volatility-adjusted naïve diversification

Assume now further, that instead of allocating an amount of wealth, the investor allocates a given amount of volatility to these positions. I.e. the investor has a total volatility budget and just allocates a volatility amount to each position (by adjusting the loan), such that the sum of position volatilities equals the total volatility budget. This sum is of course not the actual portfolio volatility (unless all assets are perfectly correlated in which case diversification is not necessary). Volatility is used with this approach only as a unit for allocation. To add some intuition, recall that the investor assumes expected excess returns per unit of volatility are the same and allocating volatility could hence be viewed as “purchasing” units of expected excess returns – i.e. each unit of volatility “spent” could be viewed as the price for one unit expected excess return.

The only difference to the naïve diversification case is therefore, that the position sizes earlier were expressed as weights \( w_i \), and now sizes are expressed as weighted volatilities \( w_i \sigma_i \). And as shown above, the best thing the mean-variance optimizing investor can do without being able to distinguish between assets’ expected returns, with no differences between assets’ volatilities, and with assuming all correlations of non-identical assets are the same, is to size each position equally. The investor will hence allocate the same amount of weighted volatility to each asset. The resulting portfolio, the type 1 risk parity portfolio, will minimize the portfolio variance – compared to all other portfolios with full allocation of the volatility budget. Hence, when assets’ expected returns and correlations are unknown, but volatilities are known, and a volatility budget is to be allocated fully in the described manner, the type 1 risk parity portfolio will be the minimum variance portfolio.

The actual minimum-variance portfolio

Now assume the investor knows in addition to volatilities also the correlations for all pairs of assets. They still don’t know anything about expected returns. Different to the risk parity type 1 approach, they don’t assume assets have equal Sharpe ratios, but instead that assets’ expected returns are all equal, in which case all portfolios (with full allocation to risky assets) will have the same expected return. As before, the optimal investment policy is hence to minimize portfolio variance. The formula for the corresponding portfolio weights is given in the appendix.

Note that the minimum variance portfolio has the property that marginal variances of all assets in the portfolio are equal – otherwise it would be possible to reduce portfolio variance by small reallocations between assets with a relatively high and a relatively low marginal variance in the current allocation, i.e.:

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7 The loan amount could of course be negative, i.e. the position could be an investment in the asset plus a positive cash position.
8 Alternatively, one could assume a limit for total variance, and then try to allocate a maximum number of “volatility”-units – thereby, under the given assumptions, maximizing expected return. The resulting portfolio will be the same as with the approach above.
9 For the sake of completeness, an algebraic proof, analogous to the naïve diversification case, is given in the appendix.
\[ \frac{\partial \sigma_p^2}{\partial w_i} = 2 \sigma_p \frac{\partial \sigma_p}{\partial w_j} \sigma_{ip} \quad \forall i, j \]

**Risk parity type 2 portfolio (ERC portfolio)**

Like the minimum variance portfolio, finding the ERC portfolio requires (an estimate of) the covariance matrix of available assets. However, different to the MVP which is characterized by all assets’ marginal variances and hence marginal volatilities having the same value, in the ERC portfolio all assets’ *weighted* marginal volatilities, aka “volatility contributions”, are equal.

An asset’s marginal volatility equals its covariance with the portfolio divided by the portfolio volatility:

\[ \frac{\partial \sigma_p}{\partial w_i} = \frac{\sigma_{ip}}{\sigma_p} \]

Therefore, equal weighted marginal volatilities imply equal weighted covariances between asset and portfolio for each asset:

\[ w_i \sigma_{ip} = w_j \sigma_{jp} \quad \forall i, j \]

If a riskless asset exists, and if all risky assets’ risk premiums were equal as well, the ERC portfolio would be the tangency portfolio\(^{11}\) – and hence all portfolios on the capital allocation line would be combinations of the ERC portfolio and the riskless asset.\(^{12}\)

Hence, one can argue that the reason for the ERC approach is the assumption that in efficient portfolios assets’ expected excess returns are proportional to their volatility contributions.\(^{13}\) As in efficient portfolios expected excess returns are proportional to marginal volatilities, Sharpe suggested interpreting volatility contributions as weighted implied expected excess returns (expected excess return contributions).\(^{14}\) With this interpretation, the ERC approach can be justified with the belief that in efficient portfolios all positions’ expected excess return contributions are equal.\(^{15}\)

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10 Risk Parity Type 2/ERC is Presented For Example In: Risk Parity Portfolios, Efficient Portfolios through True Diversification, Edward Qian, September 2005, Panagora Asset Management, see also the reference given in footnote 1 and the additional references given in that source.
11 The portfolio represented by the point where the capital allocation line (CAL, a tangent to the efficient frontier of risky assets only portfolios drawn from the point representing the riskless asset in a mean standard deviation diagram) touches the risky-assets only efficient frontier. The CAL is sometimes also referred to as “best possible CAL”.
12 In efficient portfolios, if a riskless asset exists, the ratio of expected excess returns to marginal volatility, and hence the ratio of weighted excess return to volatility contribution is equal for all assets, as otherwise the ratio of portfolio excess return per unit volatility could be improved by reallocation – see for example the reference given in footnote 13, p. 78 (there in marginal utility terms).
13 Marginal volatilities and hence risk contributions depend on the other assets in the portfolio (if the portfolio includes more than two assets). Therefore finding the ERC portfolio is mathematically more challenging than finding any of the portfolios discussed above. One method is given in the appendix.
Appendix

Proof that risk parity type I (inverse volatility) is variance-minimizing if correlations of non-identical assets are equal to zero:

\[ w_i = \frac{k}{\sigma_i} \]

\( k \) is the total “volatility” to be allocated divided by the number of assets.\(^\text{16}\)

i.e.:

\[ w_i \sigma_i = k \]

\[ w_i^2 \sigma_i^2 = k^2 \]

As correlations between non-identical assets are assumed to be zero:

\[ \sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 = (nk)^2 \]

If an amount \( x \) of weighted volatility is reallocated between two assets (here 1 and 2):

\[ \sigma_{p_{\text{new}}}^2 = \sum_{i=3}^{N} w_i^2 \sigma_i^2 + (w_1 \sigma_{1_{\text{new}}})^2 + (w_2 \sigma_{2_{\text{new}}})^2 = \]

\[ (n-2)k^2 + (k+x)^2 + (k-x)^2 = \]

\[ (n-2)k^2 + (k^2 + 2kx + x^2) + (k^2 - 2kx + x^2) = \]

\[ (n-2)k^2 + 2k^2 + 2x^2 = \]

\[ nk^2 + 2x^2 \]

\(^{16}\) The formula for the weights given here shows why the risk-parity-type-1 portfolio is also referred to as “inverse volatility” portfolio; see the reference given in footnote 1.
\[ \Delta \sigma_p^2 = \sigma_{p_{new}}^2 - \sigma_p^2 = \]
\[ nk^2 - nk^2 + 2x^2 = \]
\[ 2x^2 > 0 \]

So deviating slightly from the inverse volatility (risk parity type 1) portfolio will increase portfolio variance – and the reasoning given in footnote 2 can be applied analogously.

**Finding the minimum variance portfolio:**

Deriving the weights of the minimum variance portfolio (short positions allowed):

Minimizing variance = minimizing half the variance:

Second order condition:

\[ \sum_i w_i = 1 \]

Define and minimize the Lagrangian:

\[ L = \frac{1}{2} \sigma_p^2 - \lambda \left( \sum_i w_i - 1 \right) = \]

\[ \frac{\partial L}{\partial w_i} = \sum_j w_j \sigma_{ij} - \lambda = 0 \quad \forall i \]

\[ \sum_j w_j \sigma_{ij} = \lambda \quad \forall i \]

Write the first order conditions in vector/matrix notation:

\[ \Sigma w = \lambda \]

\[ w = \Sigma^{-1} \lambda \]

To solve for the Lagrange multiplier, write the elements of \( w \) as follows:

\[ w_i = \left( \sum_j \sigma_{ij}^{-1} \right) \lambda_i = \lambda_i \sum_j \sigma_{ij}^{-1} \]
with $\sigma^{-1}_{ij}$ being the elements of the inverse of the covariance matrix. Summing over all $w_i$ gives:

$$1 = \lambda_i \sum_i \sum_j \sigma^{-1}_{ij}$$

Solving for the Lagrange constant:

$$\lambda_i = \frac{1}{\sum_i \sum_j \sigma^{-1}_{ij}}$$

So the optimal weights are given by:

$$w_i = \frac{\sum_j \sigma^{-1}_{ij}}{\sum_i \sum_j \sigma^{-1}_{ij}}$$

**A method to find the risk parity type 2 portfolio:**

The weighted sum of risk contributions is the portfolio variance, and hence in an ERC portfolio, every asset’s risk contribution equals the portfolio variance divided by the number of assets:

$$w_i \sigma_{ip} = w_j \sigma_{jp} = rc \quad \forall i, j$$

$$\sum_i w_i \sigma_{ip} = \sigma_p^2$$

$$n \cdot rc = \sigma_p^2$$

$$rc = \frac{\sigma_p^2}{n}$$

One method of deriving the ERC portfolio (numerically) is hence minimizing the sum of squared differences between risk contributions and portfolio variance divided by number of assets, which should be zero in the true ERC portfolio, i.e.: 17

$$x_i = rc - w_i \sigma_{ip}$$

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17 This method has been suggested on p. 7, Maillard S., Roncalli T. and Teiletche J. (2009), The Properties of Equally Weighted Risk Contributions Portfolios, working paper (later published in the Journal of Portfolio Management).
\[ x_i = \frac{1}{n} - w_i \sigma_{ip} \]
\[ y = \sum_i x_i^2 \]

Minimization of \( y \) gives the weight vector leading to equal risk contributions.

In the special case with zero correlations between all pairs of assets, the ERC portfolio is simply the portfolio where the products of asset variances multiplied with the asset weight are equal for all positions and the optimal position leverage can be found analogously to the inverse volatility approach, to be:

\[ w_i = \frac{\sigma_{PT}^2}{n \sigma_i^2} \]

where \( \sigma_{PT}^2 \) is the targeted portfolio variance.