In the CAPM by Sharpe, Lintner, Mossin and Treynor, investors “mean-variance optimize” portfolio return over a single, uniform time period. This is under certain assumptions consistent with maximization of expected utility of wealth at the end of the period. As there are no future periods considered in the model, it implies the assumption that end of period wealth is consumed completely.\(^3\) So maximization of utility of wealth and consumption are equivalent, no distinction between the two is necessary.

The intertemporal models of asset pricing by Merton, Rubinstein and Breeden assume in turn that investors maximize utility of lifetime consumption plus a bequest.\(^4\)

Utility of consumption is assumed to be time-additive, i.e. the utility of lifetime consumption is the sum of the utility values of consumption at every point in time during the investor’s life, with these utility values each being independent of consumption at earlier or later times. The utility functions are considered to be (strictly) concave.

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\(^2\) This note is motivated by the intention to find a pricing model for tradeable risks that is consistent with the model setting in Froot and Stein: market participants (e.g. financial institutions) may have untradeable exposures and different asset return expectations (coming e.g. from active trading desks), and there is a period with a new investment opportunity after the initial period. Note that Froot and Stein suggest the tax advantage of corporate debt as one possible explanation for the assumed deadweight cost of holding capital (equity); therefore a more comprehensive approach would also include a discussion of the impact of taxes following e.g. Brennan and Elton and Gruber, however here for simplicity it is assumed that there is a flat tax rate which is equal for dividends and capital gains, and portfolio optimization is done based on net-of-tax returns.
\(^3\) A Bequest is here for simplicity considered as part of “consumption” in a broader sense.
Investors at any point in time will consume an amount of their wealth such that they are indifferent between consuming and investing a further marginal unit of wealth. I.e. optimal consumption at any point in time is characterized by the marginal utility of consumption at that point in time being equal to the marginal utility of (investing) wealth to facilitate future consumption.

Marginal utility of wealth depends on available investment opportunities and the future distribution of such investment opportunities, which depends on the stochastic processes of state variables, i.e. the risk-free rate, expected asset returns and the asset return covariance matrix. As highlighted by Breeden, optimal consumption takes into account the stochastics of these state variables.

As long as the stochastic processes for state variables and asset returns and the utility functions for future consumption and the bequest are known, the price of an asset that provides one future cash flow at one specific point in time can be derived as shown by Rubinstein and described in the following section.

Deriving the Rubinstein approach for valuing future cash flows

The condition for maximizing expected utility is that the first derivative of expected utility with respect to the weight of any asset is zero:

\[ \frac{\partial E(U)}{\partial w_i} = 0 \]

I.

Applying the “Law of the Unconscious Statistician”, which states that the expected value of the derivative of a function equals the derivative of the expected value of this function, I. can be written as:

\[ E(U_{w_i}) = 0 \]

II.

Using the chain rule:

\[ E(U_C C_{w_i}) = 0 \]

III.

Where \( U_C \) is the first derivative of utility with respect to optimal consumption, and \( C_{w_i} \) the first derivative of optimal consumption with respect to wealth.\(^6\)

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\(^5\) For reference see footnote 9 of the post titled “Conditions for Deriving...” in this blog.

\(^6\) Note that here is no distinction between different consumption goods. This can be justified similar to Breeden’s reasoning that the price index is locally valid, as in continuous time the impact on budget shares can be assumed to be small if not negligible, see Breeden, footnote 17.
The optimal consumption is a function of wealth and state variables describing the investment opportunity set. Hence, applying the chain rule again – and the partial differential $C_W$ with respect to wealth $W$:

IV. $E[U_C C_W W_{w_i} ] = 0$

Wealth is a function of initial capital and the portfolio return. Return can also be written as a function of an imaginary portfolio holding all wealth in the risk-free asset, and all risky positions being financed at the risk free rate:

V. $W = K(1 + r_p) = K + Kr + K\sum_i w_i (r_i - r)$

Hence:

VI. $W_{w_i} = K(r_i - r)$

So that:

VII. $E(U_C C_W K(r_i - r)) = 0$

From the definition of the covariance the following can be derived:

$E(U_C C_W K(r_i - r)) = \text{cov}(U_C, C_W K(r_i - r)) + E(U_C)E[C_W K(r_i - r)]$

So that:

$\text{cov}(U_C, C_W K(r_i - r)) + E(U_C)E[C_W K(r_i - r)] = 0$

Rearranging:

$K \text{cov}(U_C, C_W r_i) + E(U_C)KE[C_W (r_i - r)] = 0$

$\frac{K \text{cov}(U_C, C_W r_i) - K C_W r}{E(U_C)} = 0$

Divide by $KC_W$, and with $r_i = \frac{X_i}{P_i} - 1$, where $X_i$ is the cash flow from asset $i$ at time $t$, and $P$ the price of the asset in $t=0$:
\[
\frac{\text{cov}(U_C, X_i)}{P_i E(U_C)} + \frac{E \left( \frac{X_i}{P_i} - 1 \right)}{P_i} - r = 0
\]

\[
\frac{\text{cov}(U_C, X_i)}{E(U_C)} + E(X_i) = P_i (1 + r)
\]

\[
P_i = \frac{E(X_i) + \text{cov}(U_C, X_i) / E(U_C)}{(1 + r)}
\]

which is equation 1 in Rubinstein (1976) for the case with a single cash flow. At this price, the investor would not change the allocation in their portfolio (c. p.). At a higher price, portfolio efficiency could be improved by reducing the weight of this asset. And at a lower price increasing the weight would be advantageous.\(^7\)

**Deriving Breeden’s equation for the expected return as a function of an asset with perfect correlation with changes in optimal consumption**

Note the last equation from above can also be written as an expected return requirement:

\[
E(r_i) = \frac{E(X_i)}{P_0} - 1
\]

\[
E(X_i) = P_0 (1 + r) - \text{cov}(U_C, X_i) / E(U_C)
\]

\[
E(r_i) = r - \text{cov}(U_C, 1 + r_i) / E(U_C)
\]

\[
E(r_i) - r = -\frac{\text{cov}(U_C, r_i)}{E(U_C)}
\]

If following Breeden instantaneous returns are assumed to be normally distributed, one can apply Stein’s lemma.\(^8\)

\[
\mu_i - r = -\frac{EU_{CC}}{EU_C} \sigma_i C
\]

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\(^7\) If the cash flow is from an asset that is not yet in the portfolio, then one would have to account for the impact on the covariance with marginal utility, if the asset was included. If the position was small, the price calculated without adjusting the covariance could be used as an approximating hurdle rate. Further discussion is given in the concluding section of this note.

\(^8\) Changes in optimal consumption are normal over an instantaneous interval, as changes in wealth and state variables are assumed to be multivariate normal – see Breeden, page 268 equations 2 and 3 (two equations, as Breeden splits asset return into price and dividend return), and page 271, equation 1’.
Write consumption as function of optimal consumption with current wealth and state variables, plus the relative change in consumption \( \frac{dC}{C_0} = r_C \):

\[
C = C_0 (1 + r_C)
\]

So then:

\[
\mu_i - r = -\frac{E(U_{CC})}{E(U_C)} \text{cov}(\eta_i, r_C)
\]

And with:

\[
G = -\frac{E(U_{CC})}{E(U_C)}
\]

The expected return in an efficient portfolio can be written as:

\[
\mu_i - r = G \sigma_{iC}
\]

This holds for all assets in the portfolio, hence:

I. \[
\frac{\mu_i - r}{\sigma_{iC}} = G = \frac{\mu_j - r}{\sigma_{jC}}
\]

For all \(i, j\).

As pointed out by Breeden, if there is an asset \(x\) that is perfectly correlated with \(C\):

II. \[
\frac{\mu_i - r}{\sigma_{iC}} = \frac{\mu_x - r}{\sigma_{xC}} = \frac{\mu_x - r}{\sigma_{x}\sigma_C}
\]

III. \[
\mu_i - r = (\mu_x - r) \frac{\sigma_{iC}}{\sigma_{x}\sigma_C}
\]

Such an asset can be constructed in a market with all assets being traded and where for each state variable a perfectly correlated asset exists\(^9\), because when every state variable can be traded, all factors impacting changes in optimal consumption can be traded.

III can also be written as:

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\(9\) Note that Breeden’s risk aversion parameter \(T\) is the reciprocal of \(G\), and that Breeden continuous for the special case of constant \(T\).

\(10\) Breeden shows this formally on p. 281, and shows further that if there is no asset that is perfectly correlated with optimal consumption, because assets or portfolios that are perfectly correlated with the state variables don’t exist, the asset or portfolio with maximum correlation with optimal consumption takes the place of the asset or portfolio perfectly correlated with optimal consumption – see Breeden footnotes 7 and 8. An alternative, intuitive proof following Brito is given in the next section for the case that the investor holds non-marketable assets – if one accepts that any untradeable state variable could be modelled as a nonmarketable asset.
\[ \mu_i - r = (\mu_x - r) \frac{\rho_{iC} \sigma_i}{\sigma_x} \]

As correlation between the return of asset or portfolio \( x \) and optimal consumption \( C \) is equal to one, the correlation between \( i \) and \( x \) is equal to the correlation between \( i \) and \( C \), and hence:\(^{11}\)

\[ \mu_i - r = (\mu_x - r) \frac{\rho_{ix} \sigma_i}{\sigma_x} \]

Or:

IV. \[ \mu_i - r = (\mu_x - r) \frac{\sigma_{ix}}{\sigma_x^2} \]

And with \( \beta_{ix} = \frac{\sigma_{ix}}{\sigma_x^2} : \)

V. \[ \mu_i - r = (\mu_x - r) \beta_{ix} \]

**Non-marketable assets**

In analogy to Brito\(^ {12} \), one can argue that an investor who mean-variance optimizes their consumption, first creates a portfolio with tradeable assets (the hedge portfolio) that hedges all tradeable exposures that potentially impact optimal consumption and that are inherent in their portfolio of existing untradeable assets, and then combines their hedged (and hence consumption variance-minimizing) portfolio with the portfolio of tradeable assets (and exposures to state variables) that offers the highest expected increase in optimal consumption per unit of increase in variance of optimal consumption (the investment portfolio); i.e. with the same maxim as they would select their portfolio if they did not hold any non-marketable assets.

And hedging all tradeable exposures inherent in the portfolio of untradeable assets (and with impact on optimal consumption), together with the earlier assumption that optimal consumption is locally linear in wealth and the state variables, implies that the impact of tradable asset returns and tradeable state variable changes on consumption is uncorrelated with the impact of the hedged portfolio of untradeable assets on consumption.

\(^{11}\) To show this, one could write \( x \) as a regression function of \( C \) – which would have no error term given the correlation is +1.

This implication - and again the local linearity – leads to the increase in variance of optimal consumption due to the investment portfolio being equal to what the variance of optimal consumption would be if the investor held only the investment portfolio, without any untradeable assets.\(^\text{13}\)

Hence, the ideal investment portfolio in risky assets is the one that maximizes expected excess return per unit of what the variance of optimal consumption would be if the investor held only the investment portfolio.\(^\text{14}\)

Also, as the hedge portfolio removes any correlation of the impact of the investor’s untradeable assets’ returns on optimal consumption with the impact of tradable assets’ returns on optimal consumption, any tradable asset’s return covariance with changes in optimal consumption due the return of the investment portfolio, equals its covariance with changes of the investor’s total optimal consumption at that point in time.

Therefore in the optimal portfolio the relationships in I. from above hold as well if the investor holds untradeable assets, with the covariances in I. being the covariances with optimal consumption changes due to the investment portfolio return.

Further, as there will also be an asset or portfolio that is perfectly correlated with the impact on optimal consumption due to the return of the aggregate of the investment portfolio and any exposures to traded state variables; and as optimal consumption is locally linear in wealth and state variables, and hence it will be locally linear in this aggregate, the asset or portfolio that is perfectly correlated with this aggregate (i.e. the aggregate itself) will also be perfectly correlated with optimal consumption.

Finally, as the (only) other factors impacting optimal consumption are the unhedgeable return components of the investor’s untradeable assets (as the tradeable components have been hedged), which are uncorrelated with the portfolio of traded assets and exposures to traded state variables, the mentioned aggregate is also the portfolio of tradable exposures that has maximum correlation with optimal consumption (this portfolio is in the following referred to as MCP – for “maximum correlation portfolio”).

Equation II can hence be applied in the case of the investor holding non-marketable assets as well, with \(x\) now representing the instantaneous return of the MCP.

**Aggregation across investors**\(^\text{15}\)

\(^\text{13}\) The local linearity of optimal consumption, i.e. the change in optimal consumption being a linear function of asset returns and changes in state variables for small changes, is here sufficient, due to the continuous time setting. In a discrete time model, the same could be justified by assuming Constant Proportional Risk Aversion and hence a constant consumption rate, i.e. the investor consuming always the same percentage of available wealth (see Rubinstein p. 419).

\(^\text{14}\) The amount allocated to this investment portfolio would however be higher if the investor held no untradeable (and not perfectly hedgeable) assets.

Note further that a tradable asset may also be bought for the hedge portfolio – however in equilibrium an asset will not be priced higher or lower than justified given the market price of risk and the assets covariance with the market, otherwise investors would want to change their allocations in their investment portfolios.
From equation IV.:

$$\mu_i - r = \frac{\mu_x - r}{\sigma_x} \sigma_{ix}'$$

I.e. in an investor’s consumption-mean-variance efficient portfolio, the expected excess return per unit of covariance with the MCP equals the excess return of the MCP divided by its return variance.

Introducing the index $a$, to label individual investors in the market and rearranging above:

$$\frac{\mu_{ia} - r}{\sigma_{ix}'a} = \frac{\mu_{x'a} - r}{\sigma_{x'a}^2}$$

$$\frac{\mu_{ia} - r}{\mu_{x'a} - r} = \sigma_{ix}'a$$

This expression holds for the covariance of an asset with an individual investor’s MCP portfolio. So for the covariance of an asset with the wealth-weighted average MPC of all investors in the market, $\sigma_{ix}'$, the following must hold in equilibrium (when every investor considers their investment portfolio as efficient):

$$\sum_a W_a \frac{\mu_{ia} - r}{\sigma_{x'a}^2} = \sum_a \frac{W_a}{W} \sigma_{ix}'a$$

$$(\mu_i - r) \sum_a \frac{1}{\mu_{x'a} - r} = \sigma_{ix}'$$

With $\mu_i - r$ being the wealth-weighted average expected excess return of the asset. The last expression can be rearranged to:

VI.  

$$\mu_i - r = \sigma_{ix}' \sum_a \frac{1}{\frac{1}{\mu_{x'a} - r} \frac{\mu_{x'a} - r}{\sigma_{x'a}^2}}$$

I.e. in equilibrium an asset’s expected return is the covariance with the weighted average MCP times the harmonic mean of individual excess returns per unit of variance of the individual MCP divided by the number of investors.\textsuperscript{16}

Following Lintner, \[ \frac{1}{\sum a \frac{\mu^x_a - r}{\sigma^2_x a}} \] can hence be interpreted as “market price of risk”\textsuperscript{17}, i.e. the expected return required per unit of covariance with a market risk factor, which is here the return of the weighted average MCP.\textsuperscript{18} Note that this covariance can be adjusted arbitrarily by trading in the weighted average MCP, in other words the market risk factor is perfectly tradeable.

**Capital budgeting**

Traditional capital budgeting, using discount or hurdle rates as briefly mentioned in the section on Rubinstein’s approach, assumes that a new investment undertaken by the company has no (significant) impact on the market portfolio (or aggregate MCP, if modified to account for the insights quoted and described above).\textsuperscript{19} This implies that the covariance of the new investment with the aggregate MCP is a linear function of the size of the new investment. If the expected return of a company’s new investment is independent of the size of the exposure, the return contribution to a company’s portfolio (and hence the MCP) is also a linear function of the size of the exposure. Under these assumptions, the return contribution of a marginal unit of exposure to the new investment, and the contribution to the systematic risk of the company’s shares of a marginal unit of exposure to the new investment are both constant (i.e. independent of the size of the exposure), and applying a hurdle rate derived from VI would be sufficient.\textsuperscript{20} More complex analysis or additional conditions are needed if the investment’s expected return and its return covariance with the priced factor are not both a linear function of the size of the new investment, so that the relationship of expected excess return and covariance with the priced component is non-linear as well.\textsuperscript{21}

\textsuperscript{16} The result could as well be derived starting with individual risk aversion parameters, instead of excess returns per unit of variance of the individual MPCs – as for example described in the earlier post on Lintner’s CAPM extension in this blog.

\textsuperscript{17} Note that with homogeneous ratios of excess return to variance of individual MCPs, the market price of risk becomes that homogeneous ratio.

\textsuperscript{18} The risk factor is here a factor determining expected excess returns, and not to be understood as a factor in a factor model for realized returns.


\textsuperscript{20} Assuming the (optimal) size is given by exogenous constraints.

\textsuperscript{21} An example is given in “The Froot-Stein Model Revisited”, Annals of Actuarial Science, Vol. 1, Issue 1, March 2006, pp 37 – 47by Nils Høgh, Oliver Linton and Jens Perch Nielsen. For details see the post “Conditions for Deriving the Optimal Corporate Hedging Policy...” in this blog and the references given there.
Another example are valuation models for derivatives with non-linear payoff functions on the market portfolio or other assets that are based e.g. on the (extended) CAPM, when a linear relationship between expected return and covariance with the market can be established in an instantaneous interval (references are given in the appendix of post on Froot and Stein and Merton '98 in this blog).