Comparing Sharpe and Tint Surplus Optimization to the Capital Budgeting Approach with Multiple Investments in the Froot and Stein Framework

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Abstract

Below it is shown that full relative surplus optimization following Sharpe and Tint\(^1\) and Sharpe 2002\(^2\) leads to the same portfolio as a simplifying variation of the capital budgeting approach with multiple investments that forms a part of the risk management, capital budgeting and capital structure policy framework of Froot and Stein.\(^3\) The following first describes the Froot and Stein-based approach, and subsequently an alternative formulation of the target function for a portfolio return mean-variance optimizer (used e.g. by Sharpe 1990\(^4\)), which if modified to account for initial unchangeable exposures is similar to the target function used by Sharpe and Tint and Sharpe (2002). Moving to optimization with respect to relative surplus is then a simple rescaling exercise. The text concludes with a brief discussion of the remaining – purely terminological – differences between the described concepts.

Initial portfolio

An investor holds a portfolio including an initial positive or negative position in a riskless asset \(c\) with a risk-free rate of return labelled \(r\),\(^5\) and (positive or negative) exposures to risky assets, which have known current values but are assumed to be untradeable, i.e. these exposures cannot be changed.\(^6\) The

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3 Note that Sharpe and Tint do not explicitly mention the existence of a riskless asset (or lending and borrowing opportunity at the same risk-free rate). Sharpe 2002 uses cash as riskless asset. Absence of a riskless asset and different lending and borrowing rates will be discussed later in this blog.
4 William F. Sharpe, Capital Asset Prices With And Without Negative Holdings, Nobel Lecture, December 7, 1990
5 An existing long exposure may be a position in the asset directly, or a position in a forward on this asset (other derivatives will be discussed later in this blog). For an existing short position in a risky asset may or may not be a corresponding long position of equal size in the riskless asset – i.e. the short position in the risky asset could result from an earlier short sale or from a short position in a forward contract on the risky asset.
aggregate notional exposure\(^7\) of the initial untradeable stochastic asset and liability positions in the portfolio amounts to \(L\).\(^8\) The total initial wealth or capital \(K\) of the investor is therefore:

\[K = c + L\]

The portfolio of existing exposures is assumed to generate a future payoff \(P_L\), so that the future wealth \(W\) at the end of considered period, if no new exposures were entered, would be:

\[W = K + cr + P_L\]

The portfolio return could hence be written as:

\[\frac{W}{K} - 1 = w_c r + w_L r_L\]

with the return of the existing portfolio of risky exposures:

\[r_L = \frac{P_L}{L}\]

and weights of the risky exposures and the riskless asset being

\[w_L = \frac{L}{K}\]

and

\[w_c = \frac{c}{K}\]

respectively.\(^9\)

**New exposures**

The investor has the opportunity to take on additional, new exposures to \(n\) assets, here indexed with \(i\), where each asset provides a stochastic return \(r_i\).\(^10\) This may be done via a direct purchase or (short) sale,

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\(^7\) Note that short positions here are added with a negative sign – as opposed to Sharpe and Tint and Sharpe 2002, where liabilities have a positive sign and are subtracted from the portfolio value.

\(^8\) The symbol \(L\) here is chosen to avoid having to introduce additional symbols in a later section that stays close to the Sharpe and Tint notation – however it should be stressed here \(L\) is not the sum of liability values only, but the aggregate of the notional values of initial, unchangeable positive and negative exposures to risky assets.

\(^9\) It may be that \(L\) is zero. Then \(w_L r_L\) would have to be replaced with \(\frac{P_L}{K}\).

\(^{10}\) The returns of all risky assets are assumed to be multivariate normal. Note that in Froot and Stein the new exposures are to untradeable assets, as the optimal risk management policy derived earlier in the model implies that the institution considered in the model will not take any tradable risks, i.e. in future periods the institution cannot change these exposures. However, in
in which case the position in the riskless asset will be affected, as well as via entering long or short (zero net-worth) positions via forwards, which will provide a payoff equal to the notional position value $a_i$ times the underlying asset’s excess return (return minus the risk-free rate).\footnote{As a forward could hence be replicated with a combination of a positive or negative position in $a_i$ (positive for a long and negative for a short position) and a position with the same absolute value but opposite sign in the riskless asset, in the following the term “effective weight” is used, to emphasize that exposures created by forwards may be included.} The (effective) weight of the asset (underlying the new exposure) in the total portfolio is:

$$w_i = \frac{a_i}{K}$$

Changing the effective weight via a cash transaction or entering a forward changes the effective weight of the aggregate position in the riskless asset accordingly – for every new exposure to a risky asset with size $a_i$ there will be a position with size $-a_i$ in the riskless asset added to the portfolio. The resulting weight of the effective aggregate position in the riskless asset is $(w_c - w_i)$, and hence the portfolio return can be expressed as:

$$r_p = (w_c - w_i) r + w_L r_L + w_f r_f$$

Correspondingly, if several new positions are established:

$$r_p = \left( w_c - \sum w_i \right) r + w_L r_L + \sum w_i r_f$$

or:

I. $$r_p = w_c r + w_L r_L + \sum w_i (r_f - r)$$

And the future wealth is hence:

II. $$W = K(1 + r_p) = K + K \left[ w_c r + w_L r_L + \sum w_i (r_f - r) \right]$$

**Maximizing expected utility of (future) wealth**

The investor has a concave utility function $U$ of future wealth $W$, and wants to find the weight of every asset $i$ that maximizes their expected utility $EU(W)$ at the end of the considered period when all payoffs are realized, i.e. assets sold and forwards and liabilities settled. A necessary condition for a maximum of $EU(W)$ is that the first derivative with respect to $w_i$ equals zero:

III. $$\frac{\partial EU(U)}{\partial w_i} = 0$$

One-period expected utility maximization future reallocation is not considered, so that the result here would be the same if the new exposures were assumed to be unchangeable in the future.
Applying the “Law of the Unconscious Statistician”, which states that the expected value of the derivative of a function equals the derivative of the expected value of this function, III can be written as:\(^\text{12}\)

$$E \frac{\partial U}{\partial w_i} = 0$$

Using the chain rule:

$$E \left( \frac{\partial U}{\partial W} \frac{\partial W}{\partial w_i} \right) = 0$$

For more compact notation, write:

$$\frac{\partial U}{\partial W} = U'_W$$,

and:

$$\frac{\partial W}{\partial w_i} = W'_i$$

So that the condition for a local expected utility maximum can be expressed as:

IV. $$E(U'_W W'_i) = 0$$

By applying an equality that can be derived from the definition of the covariance:\(^\text{13}\)

$$\text{cov}(U'_W, W'_i) = E(U'_W W'_i) - E(U'_W)E(W'_i)$$,

IV. can be expressed as:

V. $$E(U'_W W'_i) = \text{cov}(U'_W, W'_i) + E(U'_W)E(W'_i) = 0$$.

The Rubinstein-Stein lemma states, that for normally distributed \(w_i\):\(^\text{14}\)

$$\text{cov}(U'_W, W'_i) = EU'_W \text{cov}(W, W'_i)$$

Applying this lemma to V. gives:

VI. $$E(U'_W W'_i) = EU'_W \text{cov}(W, W'_i) + EU'_W E(W'_i) = 0$$

\(^\text{12}\) For reference see footnote 4 of the post titled “on Froot and Stein revisited” in this blog.

\(^\text{13}\) For the derivation see for example: https://en.wikipedia.org/wiki/Algebraic_formula_for_the_variance#Generalization_to_covariance. This and all following online sources are as of September 6\(^\text{th}\) 2015.

\(^\text{14}\) The lemma is more commonly known as Stein’s lemma. However, in the form used here, the lemma is shown on p.421 in: The Valuation of Uncertain Income Streams and the Pricing of Options Author(s): Mark Rubinstein Source: The Bell Journal of Economics, Vol. 7, No. 2 (autumn, 1976), pp. 407-425.
The first derivative of wealth (equation II) with respect to the weight of asset $i$ is:

VII. \[ W_{wi} = K(r_i - r) \]

Writing $\mu_i$ for the expectation of the excess return $(r_i - r)$, the expected value of VII is:

VIII. \[ E(W_{wi}) = K\mu_i \]

From II and VII the covariance of wealth and its first derivative with respect to the weight of $i$ is:

IX. \[ \text{cov}(W, W_{wi}) = K^2 \text{cov}(r_i, r) = K^2 \sigma_{ip} \]

Rewriting VI with VIII and IX:

\[ EU_{W}{W}K^2\sigma_{ip} + EU_{W}K\mu_i = 0 \]

Solving for $\mu_i$ gives the optimality condition (from which further below the optimal weights will be derived):

X. \[ \mu_i = KG\sigma_{ip} \]

With

\[ G = -\frac{EU_{WW}}{EU_W} \]

This condition was first derived by Mark Rubinstein.\(^{16}\) Note that X is the condition for a local optimum, however the concavity of the utility function ensures that the portfolio meeting X is the global expected utility maximum.\(^ {17}\) The risk aversion parameter $G$, aka Rubinstein’s measure of absolute risk aversion,\(^ {18}\) may or may not be constant (or vary itself with changing portfolio composition\(^ {19}\)), depending on the

\(^{15}\) $W_{wi}$ is the change of the future portfolio value, if the weight of asset $i$ is increased marginally — expressed per measurement unit of weight — which is 1 (i.e. 100%).

\(^{16}\) See equation 6 page 614 in: A Comparative Statics Analysis of Risk Premiums, Mark E. Rubinstein, The Journal of Business, Vol. 46, No. 4 (Oct., 1973), pp. 605-615. For the slight differences to equation X. see the post “Treynor-Black...” in this blog. Note that Rubinstein in the above source derives the equation with a Taylor series expansion, while in a later source, An Aggregation Theorem for Securities Markets, Journal of Financial Economics, October 1974, he apparently uses the Rubinstein-Stein Lemma to derive the equation (there equation 16, apparently derived from equation 12). Also on page 614 in Comparative Statics..., Rubinstein points out that the equation holds for a quadratic utility function as defined by Mossin as well. The same equation can also be used by the quadratic suggested by Markowitz and Levy, which is in a relatively wide range a very good approximation of other utility functions. Further details, proofs and references will be provided later in this blog.

\(^{17}\) A proof will be given later in this blog.

\(^{18}\) The name was given to this measure by Li and Ziemba — see the post on the Li-Ziemba approximation in this blog.

\(^{19}\) See in this context the discussion on investments influencing the value of $G$ in Froot and Stein, p.67. Further, as discussed in the post Treynor-Black..., if the portfolio is efficient, the risk aversion parameter is equal to the ratio of expected return to
utility function. In case of negative exponential utility it is a constant,\(^{20}\) and equal to the Arrow-Pratt measure.\(^{21}\)

Writing now \(A\) for the aggregate of new exposures and the initial position in the riskless asset:\(^{22}\)

\[
A = \sum_{i} a_i + c = K - L = c
\]

The weight of this aggregate in the total portfolio is:

\[
w_A = \frac{A}{K} = \frac{w_c + \sum w_i}{w_c}
\]

And the payoff of the aggregate is:

\[
P_A = \sum_{i} a_i (\eta_i - r) + cr
\]

Dividing \(P_A\) by \(A\) gives the return of the aggregate:

\[
r_A = \frac{P_A}{A} = \sum_{i} a_i (\eta_i - r) + cr
\]

With \(w_A\) and \(r_A\) defined like this, equation I can be written as:

\[
r_p = w_A r_A + w_L r_L
\]

So that:

\[
\sigma_p = w_A \sigma_{iA} + w_L \sigma_{iL}
\]

with the covariances: \(\sigma_{iA} = \text{cov}(\eta_i, r_A)\) and \(\sigma_{iL} = \text{cov}(\eta_i, r_L)\).

\(X\) can then be written as:\(^{23}\)

\[
\mu_i = KG(w_A \sigma_{iA} + w_L \sigma_{iL})
\]

or:

\[\text{variance} - \text{which depends on weights, variances and covariances. See also footnote 5 in Sharpe 1990, where Sharpe hence refers to the risk tolerance parameter as being the risk tolerance parameter in case of optimal holdings.}\]

\(^{20}\) As Sharpe pointed out in Sharpe 1990. A proof of this, following Sargent, can be found at: \(\text{http://www.tau.ac.il/~spiegel/teaching/corpfin/mean-variance.pdf}\).

\(^{21}\) Further details to be provided later in this blog.

\(^{22}\) Recall that for every new exposure \(a_i\) to a risky asset, there will be a position - \(a_i\) in the riskless asset added to the portfolio and included in the sum in this equation. Note that \(A\) is not the sum of the values of new assets only, but the sum of all new positive and negative exposures to risky assets – see also footnote 8 above.

\(^{23}\) If \(L=0\) the covariance of \(\eta_i\) with the payoff \(P_L\) divided by \(K\) is to be used here instead.
Comparing XIV and X shows that portfolio optimization taking initial untradeable exposures into account is equivalent to the portfolio optimization of an investor without initial exposures and capital $A$, if $L G \sigma_{iL}$ is subtracted from the expected return of every asset $i$, as pointed out by Froot and Stein.\footnote{See Froot and Stein p. 66-67.} This may be particularly helpful in cases like those with negative exponential utility function, where the risk aversion measure is a constant or in applications where it can be assumed to be approximately constant, e.g. in stages of an investment process during which relatively small variations in portfolio composition are to be analyzed.\footnote{See e.g. Sharpe’s web site: http://web.stanford.edu/~wfsharpe/mia/rr/mia_rr2.htm#indifference and http://web.stanford.edu/~wfsharpe/mia/rr/mia_rr2.htm#negative} In general though, $G$ may vary significantly with the portfolio allocation, so that the terms to be subtracted from expected returns are not known before the portfolio optimization is completed.\footnote{See Froot and Stein p.67, where dependencies of investment decisions due to their impact on $G$ are discussed. To find the optimal portfolio in cases with $G$ varying significantly with portfolio composition, one could follow the approach by Brito and find the variance-minimizing hedge for the existing exposures, and then add units of the portfolio of new assets that has the maximum excess return per unit of portfolio standard deviation. The optimal allocation to that portfolio would be at the point where the slope of an indifference curve in a mean-standard deviation diagram equals that ratio, see page 77 in Tobin, Liquidity Preference as Behavior Towards Risk, Review of Economic Studies, 1958. Note that monotonicity and transitivity ensure that all indifference curves have the same slope at a given standard deviation. In practice a stochastic optimization algorithm could be used, or an iterative process to find an approximate solution, as for example described on Sharpe’s web site, see http://web.stanford.edu/~wfsharpe/mia/rr/mia_rr2.htm#roles and http://web.stanford.edu/~wfsharpe/mia/rr/mia_rr2.htm#inferring.}

**Solving for optimal exposure sizes:**

From XI:

$$\sigma_{iA} = \sum_j a_j \sigma_{ij}$$

So that XIII can be written as:

$$\mu_i - LG \sigma_{iL} = G \sum_j a_j \sigma_{ij}$$

Rearranging:

**XV.**

$$\frac{\mu_i - GL \sigma_{iL}}{G} = \sum_j a_j \sigma_{ij}$$

For every asset $i$, there will be one equation like XV. In matrix notation, with $\mu$ being the excess return vector, $\Omega$ being the return covariance matrix, $a$ the vector of new exposures to risky assets, and $C_{iL}$ a vector of covariances of new assets’ returns with the existing portfolio return, this set of $n$ equations can be expressed as:
\[
\frac{\mu - GLC_i L}{G} = \Omega a
\]

Solving for the optimal new exposures to risky assets gives:

\[
a = \Omega^{-1} \frac{\mu - GLC_i L}{G}
\]

With \( \Omega^{-1} \) being the inverse of the covariance matrix \( \Omega \). \(^{27}\)

**Alternative target function**

Instead of directly maximizing expected utility, one could also use a target function that is defined as:\(^{28}\)

\[
\text{XVI. } U = E(r_p) - \frac{\sigma_p^2}{t}
\]

If the investor applies the mean-variance principle, their maximization problem can always be expressed as maximizing a function like XVI. \(^{29}\)

Here, based on equation II and the symbols introduced for excess returns, \( U \) can be written as:

\[
\text{XVII. } U = \sum_i w_i \mu_i + w_L E(r_L) + w_c r - \frac{\sigma_p^2}{t}
\]

As \( w_i \) and \( w_c \) are constants, maximizing \( U \) corresponds to maximizing \( U_1 \) defined as follows:

\[
U_1 = \sum_i w_i \mu_i - \frac{\sigma_p^2}{t}
\]

\(^{27}\) The optimal allocation to the riskless asset is determined by: \( K = c + L \sum_i a_i \) and hence: \( c = K - L - \sum_i a_i \).

In Froot and Stein, *for the portfolio of existing exposures*, payoff volatility instead of return volatility is used. In the notation here, that would mean that \( \sigma_{IL} \) would be replaced with \( \sigma_{IL}^{\text{payoff}} = L \sigma_{IL} \) and hence \( w_L \) in equation XIII. would have to be replaced with \( \frac{1}{K} \) so that the optimal exposures are: \( a = \Omega^{-1} \frac{\mu - GC^{\text{payoff}}_{IL}}{G} \). This would be equation 12 in Froot and Stein, if the components compensating for priced risk were first subtracted from the expected returns (see also footnote 3 above). Note that (excess) payoffs of new exposures in Froot and Stein are expressed per unit of \( a_i \), so that one dollar can be chosen as unit, and excess returns can be used.

\(^{28}\) See e.g. Sharpe 1990

\(^{29}\) Note that as pointed out by Sharpe, the whole efficient frontier can be constructed by maximizing XVI for different values of \( t \), see Sharpe 2002, page 78, left column. For further intuition imagine the investor sets \( \frac{\sigma_p^2}{t} \), and then maximizes the expected return – the result will be a portfolio on the efficient frontier (or the capital allocation line, if a risk-free asset exists.) In reality, \( t \), portfolio variance and expected return may have to be defined at the same time depending on the utility function – see also the brief discussion above (page 6).
Derive the first order conditions for portfolio weights that maximize $U_1$:

$$\frac{\partial U_1}{\partial w_i} = U_{1w_i} = \mu_i - \frac{2}{t} \sigma_{ip} = 0, \text{ or, with XII:}$$

$$XVIII. \quad \mu_i = \frac{2}{t} (w_A \sigma_{iA} + w_L \sigma_{iL})$$

Optimal weights can be found based on XVIII analogous to the approach described in the previous section.

As an advantage of this alternative derivation one might consider that the normal distribution assumption or quadratic utility function is not directly required for the derivation.\(^{30}\) This may be helpful where another distribution that meets the criteria for mean-variance optimization corresponding to utility optimization is assumed for portfolio returns, or where mean variance is considered a reasonable approximation although the criteria for equivalence with expected utility maximization are not fully met.

If the conditions for equivalence between expected utility maximization and mean-variance optimization are met, XI and XVIII can be set equal:

$$\frac{2}{t} (w_A \sigma_{iA} + w_L \sigma_{iL}) = KG(w_A \sigma_{iA} + w_L \sigma_{iL})$$

$$\frac{2}{t} = KG$$

so that the relationship between risk tolerance parameter and Rubinstein risk aversion measure is:

$$t = \frac{2}{KG}.$$  

**Sharpe and Tint**

In the initial Sharpe and Tint setting and in Sharpe 2002, the portfolio of initial exposures includes only liabilities, and new exposures are assumed to be all assets. Portfolio “surplus” at time $t$ is defined as $A_t - L_t$ (which corresponds to what was above termed wealth $W_t$), and “relative surplus” (in the following: $r_t$) is surplus divided by $A$:\(^{31}\)

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\(^{30}\) However if the target function is meant to be compatible with expected utility of wealth maximization, these assumptions would be necessary. Also note that other distributions besides the normal lead to equivalence between expected utility maximization and mean-variance optimization, further details and references will be given later in this blog.

\(^{31}\) The notation here differs slightly from the original, as in Sharpe and Tint the value of liabilities $L$ is expressed as a positive number and subtracted from $A$. Here instead a liability is defined as an asset of which a negative amount is held in the portfolio. Note also that in Sharpe and Tint untradeable assets get introduced later as a model extension in the same manner as liabilities. Due to the additivity of the covariance, further distinction would be straightforward – e.g. a split of (subjectively) unchangeable positions into asset classes or risk factor exposures, which may e.g. be present in discounted future sources of income.
\[
\frac{A_t + L_t}{A} = \frac{A(1 + r_A) + L(1 + r_L)}{A} = \frac{A(1 + r_A) + L(1 + r_L)}{A} = 1 + r_A + 1 + \frac{L}{A} = r_s
\]

Similar to XVII one can define a “utility” function with respect to relative surplus:

\[
XIX. \quad U_{s1} = E(r_s) - \frac{\sigma_s^2}{t_s} = 2 + E(r_A) + \frac{L}{A}E(r_L) - \frac{\sigma_A^2 + 2 \frac{L}{A} \sigma_{AL} + \left( \frac{L}{A} \right)^2 \sigma_L^2}{t_s}
\]

As the initial liabilities \(L\) and asset value \(A\) are constants, maximizing \(U_{s1}\) is equivalent to maximizing: \(^{32}\)

\[
XX. \quad U_s = E(r_A) - 2 \frac{L}{At_s} \sigma_{AL} - \frac{\sigma_A^2}{t_s}
\]

This is the objective function in Sharpe 2002 (equation 16), which is a special case of the final objective function in Sharpe and Tint. \(^{33}\)

**Equivalence of full surplus optimization and portfolio return mean variance optimization:**

As mentioned above, \(A+L=W\) so that \(\frac{A+L}{K} = \frac{W}{K} = r_p + 1\), and the relative surplus can also be written as:

\[
XXI. \quad r_s = (r_p + 1) \frac{K}{A}
\]

With expected value:

\[
\mu_s = [E(r_p) + 1] \frac{K}{A}
\]

And variance:

\(^{32}\) \(L, A\) are here the values of assets and liabilities before any new positions are entered. As there are no constraints on short positions in risky assets and or the riskless asset, the size of assets and liabilities can of course change with new positions.

\(^{33}\) In Sharpe and Tint, surplus includes the “importance” parameter \(k\): \(S_t = A(1 + r_A) + KL(1 + r_L)\). As \(r_t\) is here multiplied by \(k\), \(\sigma_{AL}\) has to be multiplied by \(k\) as well. Hence, the target function including \(k\) is: \(U_{ST} = \mu_A - \frac{\sigma_A^2}{t_s} - 2 \frac{KL}{t_s} \sigma_{AL} \), which is the original Sharpe and Tint objective function (Sharpe and Tint p. 7 left column), except the sign for liabilities (see above footnotes 7 and 31). Note further that Sharpe and Tint name the right hand term, \(LHC_A\), the “liability hedging credit, \(LHC_A\). The target function to be maximized is then: \(\mu_{A+LHC} = \mu_A - \frac{\sigma_A^2}{t_s} \), Note that \(LHC_A = -\frac{2L}{t_s} \sum_i q_i \sigma_i L\), so that with defining \(LHC_i = -\frac{2L}{t_s} \sigma_i L\), one gets: \(LHC_A = \sum_i q_i LHC_i\). With constant risk tolerance, the target function would be maximized if the \(LHC_i\) is added to each asset’s return and a mean variance optimization is performed using mean returns adjusted in this manner (Sharpe and Tint p. 9) – similarly to the adjustment mentioned above (see page 6 and the reference given in footnote 23).

Note further, that the “importance” parameter \(k\) can hence be interpreted as weight given to \(-\frac{2L}{t_s} \sigma_i L\) relative to the case of full surplus optimization (full surplus optimization means \(k=100\%\), i.e. the liability hedging credit has the same weight as the expected asset return), or relative to the weight of asset \(i\)’s expected excess return.
\[ \sigma_s^2 = \left( \frac{K}{A} \right)^2 \sigma_p^2 \]

Hence, the utility function based on relative surplus in XX. can also be written as:

XXII. \[ U = \left[ E(r_p) + 1 \right] K A - \left( \frac{K}{A} \right)^2 \frac{\sigma_p^2}{t_s} \]

After defining the risk tolerance parameter for surplus risk\(^{34}\) as a function of the risk tolerance parameter for portfolio return variance consistently with the expression for relative surplus in XXI:

XXIII. \[ t_s = t \frac{K}{A}, \]

XXII. can be written as:

XXIV. \[ U = \frac{K}{A} \left[ E(r_p) + 1 - \frac{\sigma_p^2}{t} \right] \]

And as K and A are constants maximizing XXIV. corresponds to maximizing:

XXV. \[ U = E(r_p) - \frac{\sigma_p^2}{t}, \]

which is exactly the same utility function as XVII. Hence, the optimality condition must be the same as XIX. Replacing t in XIX with \( t = t_s \frac{A}{K} \) (from XXIII) gives:

\[ \mu_l = \frac{2K}{t_s A} \left( w_A \sigma_{lA} + w_L \sigma_{lL} \right) = \frac{2}{t_s} \left( \sigma_{lA} + \frac{L}{A} \sigma_{lL} \right), \]

which is equation 20 in Sharpe 2002.\(^{35}\) So with the asset and liability return as defined above, the result of Sharpe 2002, i.e. the Sharpe and Tint result for full surplus optimization, leads to the same portfolio as optimizing portfolio return, and if the conditions for equivalence of expected utility maximization and mean variance optimization are met (which are conditions needed to apply the Froot and Stein-based approach described above), this is the portfolio that maximizes expected utility of future wealth, i.e. the same portfolio that can also be found with the Froot and Stein based approach.

**Assets and liabilities vs. variable and fixed exposures**

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\(^{34}\) Sharpe 2002 uses the term “surplus risk” on p. 81.

\(^{35}\) Note that Sharpe 2002 uses the index \( p \) instead of \( A \), as \( p \) in Sharpe 2002 refers to the asset portfolio which he discussed earlier, while here it is the total portfolio.
As emphasized by Sharpe 2002\textsuperscript{36}, relative surplus optimization in Sharpe 2002 and Tint and Sharpe 2002 does not include constraints on short positions – so that “assets” could therefore also be new negative exposures to risky assets. However a negative position in an asset is nothing else but a liability – an asset owed to another party. Further, initial untradeable positions can be assets as well – hence Sharpe and Tint extend the model to have both liabilities and untradeable assets in the initial portfolio. The comparison with Froot and Stein shows that the important distinction is not between assets and liabilities, but between initial untradeable exposures and new positions to be decided about – as in their model terminology. However, there are likely applications, e.g. for pension funds, which are referred to in Sharpe and Tint and Sharpe 2002, where existing positions are only liabilities, and new positions (including in the riskless asset) are restricted to long exposures (and the optimal portfolio from the unconstrained optimization including short positions may provide a benchmark solution, but cannot be implemented perfectly)\textsuperscript{37}, and it appears Sharpe and Tint and Sharpe 2002 had such cases in mind, as opposed to Froot and Stein, who explicitly assume financed positions and mention for example a trading desk as an application that typically involves leverage.

\textsuperscript{36} p. 78 and 84.
\textsuperscript{37} Ibid.