Proof of the Adjusted Basic Fundamental Law of Active Management

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The excess return, \( a \), of an asset above a chosen benchmark\(^1\) can be split into two components: part \( b \), the direction of which can be forecasted by the portfolio manager once they have received proprietary information (e.g. a signal from a trading model), and part \( c \) that cannot be forecasted at all:

I. \[ a = b + c \]

c is assumed to be normally distributed with an expected value of zero; from the perspective of an uninformed market participant, the same holds for \( b \). \( b \) and \( c \) are independent of each other.

Depending on their forecast for \( b \) being positive or negative, the portfolio manager enters a long or short position in the asset, and a position in the benchmark in the opposite direction of the position in the asset with the same notional exposure. The PM will therefore realize a profit of \( a \) per dollar exposure (return) if the position in the asset is long, or \(-a\) if the position is short. With regards to the forecastable part, the PM is always correctly positioned, and the return of the position, labelled \( d \), is therefore:

II. \[ d = |b| \pm c \]

As \( b \) is normally distributed, \(|b|\) has a half-normal distribution, with mean \( E(|b|) \) and variance \( \sigma_b^2 \) given as follows:\(^2\)

III. \[ E(|b|) = \sigma_b \frac{\sqrt{2}}{\pi} \]

IV. \[ \sigma_{|b|}^2 = \sigma_b^2 \left(1 - \frac{2}{\pi}\right) \]

Because \( \sigma_c^2 = \sigma_a^2 - \sigma_b^2 \) and expressing the correlation of \( a \) and \( b \) as \( \rho = \frac{\sigma_b}{\sigma_a \sigma_b} = \frac{\sigma_b}{\sigma_a} \), it follows from II. with IV for the variance of \( d \):

V. \[ \sigma_d^2 = \sigma_a^2 - \frac{2 \rho \sigma_a^2}{\pi} = \sigma_b^2 \left(1 - \frac{2}{\pi}\right) \]

With \( E(c) = 0 \), it follows for \( \alpha = E(d) \):

\[ \alpha = \sigma_b \frac{\sqrt{2}}{\pi} \]

Grinold and Kahn assume for the derivation of their fundamental law, that \( \rho \) is the same for each asset, and that any two (different) assets’ excess returns are uncorrelated. In this case, the best portfolio the

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\(^1\) this can e.g. be the risk-free rate, the rate of an index or a portfolio consisting of cash and the index with the same systematic risk as the asset

\(^2\) See https://www.randomservices.org/random/special/FoldedNormal.html for details on the half-normal distribution.
PM can build is achieved by first levering up (or down) all assets to have the same $\sigma_a$ (and hence also the same $\alpha$), and then allocating the same weight to each of these scaled assets – i.e. the optimal portfolio is the risk parity portfolio.  

With $n$ assets it follows:

$$IR_p^2 = \frac{\alpha_p^2}{\sigma_p^2} = \frac{\sigma^2}{\pi} \frac{2}{\alpha^2 (1 - \frac{2}{\pi})} = \frac{2n}{\sigma^2 - 2}$$

Labelling $n = BR$ (for Breadth) and $\rho = IC$ (information coefficient) like Grinold and Kahn, suppressing the $p$ index and rearranging gives the final formulation of the Adjusted Basic Fundamental Law of Active Management:

$$IR = IC \sqrt{BR \frac{2}{\pi - 2IC^2}}$$

This result differs from the original (Basic) Fundamental Law, which ignores the reduction in variance due to the knowledge of the sign of the forecastable return component. Both formulations have in common, that a higher IC and higher Breadth will c.p. lead to a higher IR. However, with both IC and Breadth of strategies or portfolios differing, the ratios may lead to a different ranking of these portfolios or strategies.

Appendix: proof that the risk-parity portfolio is the efficient portfolio when all assets are uncorrelated with each other and have the same expected return and return variance

Recall that when all assets under consideration have the same expected return, the mean-variance efficient portfolio is the minimum variance portfolio. And when all assets are uncorrelated with each other, and each has the same variance $\sigma^2$, the portfolio variance is given by: $\sigma_p^2 = \sigma^2 \sum w_i^2$, so that minimizing $\sum w_i^2$ corresponds to minimizing the portfolio variance; and if all weights are equal, any change to the allocation will increase $\sum w_i^2$ – consider for example a reduction in the weight of asset 1 by a small amount $x$ and a corresponding increase of the weight of asset 2 by the same amount – the sum of the squares of the weights is then given by:

$$[(w_1 - x)^2 + (w_2 + x)^2] - w_1^2 - w_2^2 = 2w_2x - 2w_1x + 2x^2.$$  In the initial, equal allocation with $w_1 = w_2$, this equals $2x^2$, which is positive for any non-zero $x$ (and hence deviation from the initial, equal allocation would cause an increase in portfolio variance).

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3 See the proof in the appendix.
5 For the sake of completeness, the equation shows also that if the allocation is non-equal, then increasing the deviation, i.e raising the weight of an asset with a relatively high weight ($w_1$) and decreasing the weight of an asset with a relatively low weight ($w_2$) would further increase variance, because then $2w_1x > 2w_2x$, so to reduce variance starting from any non-equal weight allocation, one would have to move closer to the equal-weight allocation.